3. Operators on a Hilbert Space.

A Hilbert space $\mathcal{H}$ is a vector space over the real or complex scalars endowed with an inner product $\langle \, , \, \rangle$ than maps $\mathcal{H} \times \mathcal{H}$ into $\mathbf{R}$ or $\mathbf{C}$ that satisfies the following properties.

1. $\langle x, y\rangle = \overline{\langle y, x\rangle}$ and $\langle x, y\rangle$ is linear in x, i.e. $\langle a_1x_1 + a_2x_2, y\rangle = a_1\langle x_1, y\rangle + a_2\langle x_2, y\rangle$ and semilinear in y, that is $\langle x, a_1y_1 + a_1y_2\rangle = \overline{a_1}\langle x, y_1\rangle + \overline{a_2}\langle x, y_2\rangle$

2. $\langle x, x\rangle \geq 0$ and is equal to 0 if and only if $x = 0$. It follows that $\|x\| = \langle x, x\rangle^{\frac{1}{2}}$ is a norm and

3. $\mathcal{H}$ is complete under this norm, as a mertic space with $d(x, y) = \|x - y\|$.

We first note that $\langle ax + by, ax + by\rangle = \|a\|^2\langle x, x\rangle + \|b\|^2\langle y, y\rangle + 2RPa\bar{b}\langle x, y\rangle \geq 0$ for all values of a and b. This forces

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle$$

and

$$\|x + y\| \leq \|x\| + \|y\|$$

for all $x, y \in \mathcal{H}$ This makes $d(x, y) = \|x - y\|$ in to a metric and $\mathcal{H}$ is assumed to be complete under this metric.

Example 1. $\mathcal{H} = L_2[0, 1]$. $\langle f, g\rangle = \int_0^1 f(s)\overline{g(s)}ds$

Example 2. $\mathcal{H} = l_2[Z^+]$. $\langle\{a_n\}, \{b_n\}\rangle = \sum_{n=1}^{\infty} a_n\overline{b_n}$

We say that x and y are orthogonal or $x \perp y$ if $\langle x, y\rangle = 0$. A collection $\{x_\alpha\}$ is mutually orthogonal if $\langle x_\alpha, x_\beta\rangle = 0$ for $\alpha \neq \beta$. It is an orthonormal family if in addition $\|x_\alpha\| = 1$ for every α. Any two vectors in an orthonormal family are at a distance $\sqrt{2}$. In a separable Hilbert space any orthonormal set is either finite or countable. A maximal collection of orthonormal $\{e_\alpha\}$ vectors in $\mathcal{H}$ is a basis and

$$x = \sum_\alpha \langle x, e_\alpha\rangle e_\alpha$$

is a convergent expansion with

$$\|x\|^2 = \langle x \, , \, x\rangle = \sum_\alpha |\langle x, e_\alpha\rangle|^2$$

For any subspace $\mathcal{K} \subset \mathcal{H}$ there is the orthogonal complement $\mathcal{K}^\perp = \{y : y \perp \mathcal{K}\}$. $(\mathcal{K}^\perp)^\perp = \mathcal{K}$. $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}^\perp$. If $\Lambda(x)$ is a bounded linear functional on $\mathcal{H}$ there is a unique $y \in \mathcal{H}$ such that $\Lambda(x) = \langle x \, , y\rangle$. To prove it let us look at the null space $\mathcal{K} = \{x : \Lambda(x) = 0\}$. It has codimension 1 and has x_0 that is orthogonal to $\mathcal{K}$ and $\|x_0\| = 1$ with $\Lambda(x_0) = c \neq 0$. Claim $\Lambda(x) = \langle \, x, \, \bar{c}x_0\rangle$. True on $\mathcal{K}$ and true for $x = x_0$. They span $\mathcal{H}$.

Weak topology. $x_n \hookrightarrow x$ if $\langle\, y,\, x_n\rangle \to \langle\, y,\, x\rangle$ for all $y \in \mathcal{H}$. The unit ball $\{x : \|x\| \le 1\}$ is compact in the weak topology. That is, given any bounded sequence x_n with $\|x_n\| \le C$ there is a sub sequence $x_{n_j} \hookrightarrow x$. To see this we can assume $\mathcal{H}$ is separable. It is enough to check it for a countable dense set of $y \in \mathcal{H}$. But for each y, $\langle\, y,\, x_n\rangle$ is bounded and we can extract a subsequence x_{n_j} such that $\langle y\,,\, x_{n_j}\rangle$ has a limit. Diagonalization works. We get a subsequence that works for a countable dense set and hence for all y. The limit is a bounded linear functional of y and is $\langle\, y,\, x_0\rangle$ for some $x_0 \in \mathcal{H}$

Orthogonal Projection. If $\mathcal{K} \subset \mathcal{H}$ then $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}^{\perp}$ and x can be uniqulely decomposed as $x = x_1 + x_2$ with $x_1 \in \mathcal{K}$ and $x_2 \in \mathcal{K}^{\perp}$. The maps $P_i : x \to x_i$ are self adjoint, satisfy $P_i^2 = P_i$, $P_1 P_2 = P_2 P_1 = 0$ and $P_1 + P_2 = I$. The infimum $\inf_{y \in \mathcal{K}} \|y - x\|$ is attained when $y = P_1 x$.

Problem. 1. If $x_n \hookrightarrow x$ then $\|x\| \le \liminf_{n\to\infty} \|x_n\|$. If $x_n \hookrightarrow x$ and $\|x_n\| \to \|x\|$ then $\|x_n - x\| \to 0$.

Linear Operators on $\mathcal{H}$. A map T from one Hilbert space $\mathcal{H}$ to another Hilbert space $\mathcal{K}$ is a bounded linear operator if it is linear i.e. $T(ax + by) = aTx + bTy$ and bounded i.e. $\|Tx\| \le C\|x\|$. A linear map is continuous if and only if it is bounded. $\|T\| = \sup_{\|x\|\le 1} \|Tx\|$. $\|T_1 T_2\| \le \|T_1\|\|T_2\|$. A linear operator T is compact if the image under T of the unit ball $\|x\| \le 1$ compact in $\mathcal{K}$. The adjoint T^* of a bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ is defined by $\langle T^* x, y\rangle = \langle x, Ty\rangle$. One checks that $(aT_1 + bT_2)^* = \bar{a}_1 T_1^* + \bar{a}_2 T_2^*$ and $(T_1 T_2)^* = T_2^* T_1^*$. An operator T is self adjoint if $T^* = T$ i.e. $\langle Tx, y\rangle = \langle x, Ty\rangle$. In general the product $T_1 T_2$ of two self adjoint operators is not self adjoint unless they commute, i.e $T_1 T_2 = T_2 T_1$. If T is self adjoint so is any $p(T)$ for any polynomial p with real coefficients.

The resolvent set of an operator T in Hilbert Space over the complex numbers is $z \in \mathbf{C}$ such that $(zI - T)^{-1}$ exists as a bounded operator., i.e. $(zI - T)$ is one to one, onto and (therefore has a bounded inverse), its complement is the spectrum $\mathbf{S}(T)$.

If $z \in \mathbf{S}(T)$ then $|z| \le \|T\|$. If $|z| > \|T\|$,

$$(zI - T)^{-1} = z^{-1}(I - \frac{T}{z})^{-1} = \sum_{n\ge 0} \frac{T^n}{z^{n+1}}$$

exists as a bounded operator and so $z \notin \mathbf{S}(T)$. If $\mathbf{S}(T)$ is empty $(zI - T)^{-1}$ is entire and tends to 0 at ∞. Therefore $(I - \frac{T}{z})^{-1} \equiv 0$. Cannot be!

If $zI - T$ may not be invertible because it has a null space i.e nontrivial solutions exist for $Tx = zx$ where z is a complex scalar. Then $z \in \mathbf{S}(T)$ and z is an eigenvalue with x as the eigenvector.

If T is a self-adjoint operator $\mathbf{S}(T) \subset [-\|T\|, \|T\|] \subset \mathbf{R}$. It is enough to show $z = a + ib \notin \mathbf{S}(T)$ if $b \ne 0$.

Problem. 2. Show that for any bounded operator T, if $\mathbf{N}(T) = \{x : Tx = 0\}$ is the null space and $\mathbf{R}(T) = \{y : y = Tx\}$ for some x is the range then $\mathbf{N}(T^*) = \overline{\mathbf{R}(T)}$.

To prove $z = a + ib \notin \mathbf{S}(T)$ it is enough to show that $Tx = zx$ has no nonzero solution and that $\mathbf{R}(T - zI)$ is closed.Then it can not be a proper subspace because then the orthogonal complement which is the null space of $T^* - zI = T - zI$ would be nontrivial. We next need to prove that the range is dense. An inequality of the form $\|(T - zI)x\| \geq c\|x\|$ is enough, because if $y_n = (T - zI)x_n$ has a limit y then x_n will be a Cauchy sequence with a limit x and $(zI - T)x = y$.

$$\begin{aligned}
\langle (zI - T)x, (zI - T)x \rangle &= \|a\|^2\|x\|^2 + \|b\|^2\|x\|^2 + \|Tx\|^2 - \langle (a + ib)x, Tx \rangle - \langle Tx, (a + ib)x \rangle \\
&= \|a\|^2\|x\|^2 + \|b\|^2\|x\|^2 + \|Tx\|^2 - (a + ib)\langle Tx, x \rangle - (a - ib)\langle Tx, x \rangle \\
&= \|a\|^2\|x\|^2 + \|b\|^2\|x\|^2 + \|Tx\|^2 - 2a\langle Tx, x \rangle \\
&= \|b\|^2\|x\|^2 + \|Tx - ax\|^2 \\
&\geq \|b\|^2\|x\|^2
\end{aligned}$$

An operator $T : \mathcal{H} \to \mathcal{K}$ is completely continuous or compact if any bounded sequence x_n has a subsequence x_{n_j} such that Tx_{n_j} converges. In other words the image under T of the unit ball $\|x\| \leq 1$ in $\mathcal{H}$ is compact in $\mathcal{K}$ Often $\mathcal{K} = \mathcal{H}$.

An eigenvalue λ of an operator T from $\mathcal{H} \to \mathcal{H}$ is one for which $Tx = \lambda x$ has a nontrivial solution and the corresponding x is the eigenvector.

Theorem. Let A be a self adjoint compact operator from $\mathcal{H} \to \mathcal{H}$. Then there are eigenvalues and eigenspaces

$$E_\lambda = \{x : Ax = \lambda x\}$$

that are nontrivial only for a countable set $\{\lambda_j\} \subset R$ such that for $\lambda_j \neq 0$, E_{λ_j} are finite dimensional and the only point of accumulation of $\{\lambda_j\}$ is 0. E_0 itself can be trivial, or nontrivial of finite or infinite dimension. $\{E_{\lambda_j}\}$ are mutually orthogonal and

$$\mathcal{H} = \oplus E_{\lambda_j}$$

Proof. Let $\lambda = \sup_{\|x\| \leq 1} \langle Ax\,,\, x \rangle$. Clearly $\lambda \geq 0$ and assume that $\lambda > 0$. There is a sequence x_n with $\|x_n\| \leq 1$ and $\langle Ax_n\,,\, x_n \rangle \to \lambda$. Choose a subsequence x_{n_j} that converges weakly to x_0. Then Ax_{n_j} must converge strongly (in norm) to Ax_0. Implies $\langle Ax_{n_j}, x_{n_j} \rangle \to \langle Ax_0, x_0 \rangle = \lambda$. If $\|x_0\| = c < 1$, $\langle Ac^{-1}x_0, c^{-1}x_0 \rangle = c^{-2}\lambda > \lambda = \sup_{\|x\| \leq 1} \langle Ax, x \rangle$. A contradiction. So $\|x_0\| = 1$ and the supremum is attained at x_0. In particular for $y \perp x_0$

$$F(\epsilon) = \frac{1}{1 + \epsilon^2} \langle Ax_0 + \epsilon y, x_0 + \epsilon y \rangle \geq \lambda = F(0)$$

It follows that $F'(0) = \langle Ax_0, y \rangle = 0$. If $Ax_0 \perp y$ whenever $x_0 \perp y$, $Ax_0 = cx_0$ and $c = \langle Ax_0, x_0 \rangle = \lambda$. We can repeat the process on $\mathcal{K} = \{y : y \perp x_0\}$ and proceed to get a sequence of eigenvalues $\lambda_n > 0$, with mutually orthogonal eigenvectors x_n satisfying $\|x_n\| = 1$ and $Ax_n = \lambda_n x_n$. The process may send at a finite stage are proceed without end. We note that if $\|x_n\| = 1$ and $\{x_n\}$ is mutually orthogonal

$$\sum_n |\langle y, x_n \rangle|^2 \leq \|y\|^2$$

and $x_n \hookrightarrow 0$. $\|Ax_n\| \to 0$ and $\lambda_n \to 0$. If $\mathcal{K}^+$ is the span of $\{x_n\}$, then on $\mathcal{K}^\perp$, $\langle Ax, x\rangle \leq 0$. We repeat the process with $-A$ and recover negative eigenvalues and eigenvectors corresponding to them, the eigenvectors span $\mathcal{K}^-$ forcing $A = 0$ on $[\mathcal{K}^+ \oplus \mathcal{K}^-]^\perp$.

A self adjoint operator T is positive semidefinite, i.e. $(T \geq 0)$ if $\langle Tx\,,\, x\rangle \geq 0$ for all $x \in \mathcal{H}$.

Theorem If T is a self adjoint operator and if $p(t)$ is a polynomial with real coefficients such that $p(t) \geq 0$ on the interval $[-\|T, \|T\|]$ then $p(T)$ is positive semi definite.

The proof proceeds along these steps.

If $A \geq 0$, there is a selfadjoint operator $B \geq 0$ that commutes with A, is in fact a limit of polynomials of A such that $B^2 = A$. By multiplying by a constant we can assume that $0 \leq A \leq I$. Then since

$$\sqrt{\lambda} = \sqrt{1-(1-\lambda)} = 1 - \frac{1}{2}(1-\lambda) - \sum_{n\geq 2} \frac{1\cdot 3\cdot (2n-3)}{2^n n!}(1-\lambda)^n$$

the series

$$\sum_{n\geq 2} \frac{1\cdot 3\cdot (2n-3)}{2^n n!}$$

converges,

$$B = \sqrt{A} = \sqrt{1-(1-A)} = 1 - \frac{1}{2}(1-A) - \sum_{n\geq 2} \frac{1\cdot 3\cdot (2n-3)}{2^n n!}(1-A)^n$$

is well defined, is a self adjoint operator, commutes with A is a limit in operator norm of polynomials in A and $B^2 = A$. If $A_1 \geq 0$ and $A_2 \geq 0$ are self adjoint operators that commute, then A_1A_2 is self-adjoint and $A_1A_2 \geq 0$. $A_i = B_i^2$ for $i = 1, 2$. They all mutually commute and $A_1A_2 = (B_1B_2)^2 \geq 0$.

Let the roots of $p(t) = 0$ be $\{t_j\}$. They come in different types. Complex pairs $\{a_j \pm ib_j\}$ $\{c_j \leq -\|T\|\}, \{d_j \geq \|T\|\}$ and roots of even multiplicity $\theta_j \in (-\|T\|, \|T)$. For some $c > 0$

$$p(t) = c\Pi(t-\theta_j)^{2n_j}\Pi(t-a_j)^2 + b_j^2)\Pi(t-c_j)\Pi(d_j - t)$$

and

$$p(T) = c\Pi(T-\theta_j I)^{2n_j}\Pi[(T-a_jI)^2 + b_j^2 I]\Pi(T-c_jI)\Pi(d_jI - T) \geq 0$$

Remark. If f is a continuous function on $[-\|T\|, \|T\|]$, it is a uniform limit of polynomials $p_n(t)$ and then $p_n(T)$ will have a limit $f(T)$. This defines $f(T)$ for $f \in C([-\|T\|, \|T\|)$.

$$\|f(T)\| \leq \sup_{-\|T\|\leq t\leq \|T\|} |f(t)|$$

The linear functional $\langle f(T)x, x\rangle$ is a nonnegative linear functional having a representation

$$\Lambda_x(f) = \int_{[-\|T\|,\|T\|]} f(t)\mu_{(x,x)}(dt)$$

where $\mu_{(x,x)}$ is a nonnegative measure of mass $\|x\|^2$ supported on $[-\|T\|, \|T\|]$. We define

$$\mu_{(x,y)} = \frac{1}{4}[\mu_{(x+y,x+y)} - \mu_{(x-y,x-y)}]$$

in the real case and in the complex case

$$\mu_{(x,y)} = \frac{1}{4}[\mu_{(x+y,x+y)} - \mu_{(x-y,x-y)} - i\mu_{(x+iy,x+iy)} + i\mu_{(x-iy,x-iy)}]$$

Now $\int f(t)\mu_{(x,y)}(dt) = \langle f(T)x, y\rangle$ is defined for all bounded measurable functions f. Satisfies $(fg)(T) = f(T)g(T)$.

$$\langle f(T)g(T)x, y\rangle = \int f(t)g(t)\mu_{(x,y)}(dt)$$

Pass to the limit from polynomials. Use bounded convergence theorem on the right and weak limits on the left.

Problem 3. Show that for any $x \in \mathcal{H}$, $\mu_{(x,x)}[(\mathbf{S}(T))^c] = 0$

Hint: Prove it first when $\mathbf{S}(T) \subset \{\lambda : |\lambda| \geq \ell\}$ for some ℓ and then show that it is enough.

Problem 4. Identify the spectral measures $\mu_{(x,x)}(dt)$ for a compact self-adjoint operator A.

Projection valued measures. If $E \subset [-\|T\|, \|T\|]$ is a Borel set then $\chi_E(T)$ is well defined. $\langle \chi_E(T)x, y\rangle = \int_E \mu_{(x,y)}(dt)$. Since $\chi_E^2 = \chi_E$, $\sigma(E) = \chi_E(T)$ is a projection. $\sigma(E)$ is a projection valued measure. It satisfies

1. For any $E \in \mathcal{B}$, $\sigma(E)$ is an orthogonal projection.

2. For disjoint Borel sets $\{E_i\}$, $\sigma(E_i)\sigma(E_j) = 0$ for $i \neq j$, and $\sigma(\cup E_i) = \sum_i \sigma(E_i)$.

Hilbert-Schmidt Operators. An operator A on a separable Hilbert space $\mathcal{H}$ is **Hilbert-Schmidt** if for some orthonormal basis $\{e_j\}$, $\sum_{i,j} |\langle Ae_i, e_j\rangle|^2 < \infty$.

Problem 5. Prove that the definition is independent of the basis and that all Hilbert-Schmidt operators are compact.

Trace Class Operators. A positive semidefinite self adjoint operator A is of trace class if $\sum_i \langle Ae_i, e_i\rangle$ is finite for some basis. Then it is finite on any basis and Trace $A = \sum_i \langle Ae_i, e_i\rangle$ is well defined. A is Hilbert-Schmidt if and only if A^*A or equivalently AA^* is of trace class.

Problem 6. Show that if A is a compact operator, the nonzero eigenvalues of AA^* and A^*A are the same and have the same multiplicity. In particular their traces are both finite and equal or both infinite.

Consider the operator on $L_2[0,1]$,

$$(Tf)(x) = \int_0^1 f(y)k(x,y)dy$$

is well defined as a bounded operator, if $\int_0^1 \int_0^1 |k(x,y)|^2 dxdy < \infty$ and is in fact Hilbert-Schmidt. It is self adjoint if $k(x,y) = k(y,x)$ and then the eigenvalues and eigenfunctions satisfy

$$\sum_j \lambda_j^2 = \int_0^1 \int_0^1 |k(x,y)|^2 dxdy$$

$$\sum_{i,j} \lambda_j f_j(x) f_j(y) = k(x,y) \tag{1}$$

in $L_2[[0,1]^2]$. If $k(x,y)$ is continuous and positive definite (i.e. $\{k(x_i,x_j)\}$ is a positive semidefinite matrix for any finite collection $\{x_i\}$), T is positive definite operator which is trace class with trace equal to $\int_0^1 k(x,x)dx$. The convergence in (1) is uniform.

Problem 8. Consider the operator

$$(Tf)(x) = \int_0^1 f(y)k(x,y)dy$$

on $L_2[0.1]$, where $k(x,y) = \min(x,y) - xy$, Find all the eigenvalues and eigenfunctions. Deduce the value of the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$.